Linear Programming
Math 70 – 460 (Mathematical Models for Consultants)
Carnegie Mellon University - Spring 2009

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1. System of Linear Inequalities in Two Variables

Before tackling the larger matter of linear programming, it will be worthwhile to be certain that we understand how to solve systems of linear inequalities.

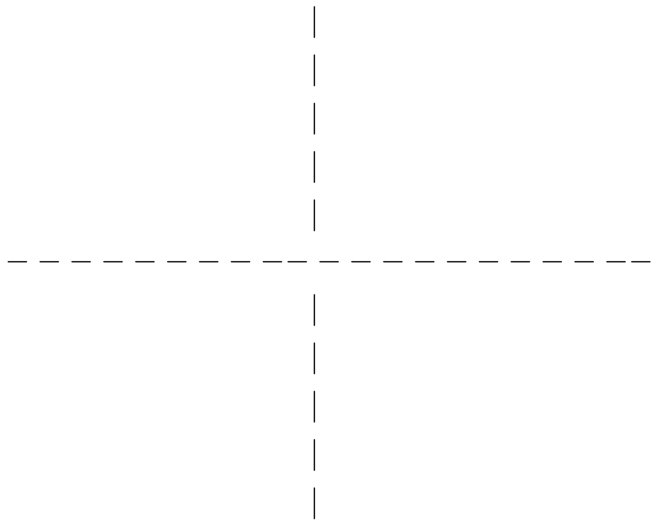
The ***solution set*** to a system of linear inequalities is the collection of all points that satisfy each inequality in the system. This is most easily represented by a graph. A solution set is considered to be ***bounded*** if the entire region can be enclosed in a circle. If the region cannot be entirely enclosed in an arbitrary circle, then the region is said to be ***unbounded***. As well, we note that a point in the solution region that is the intersection of two boundary lines is called a ***corner point***. The significance of the corner points will be discussed later.

1.1. Examples.

Solve each of the following Systems of Linear Inequalities graphically. As well, label all corner points and state whether the solution region is bounded or unbounded.

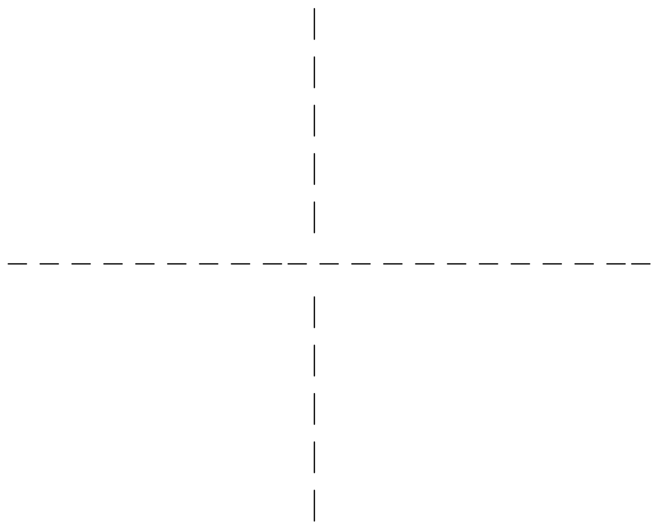
Example 1.1.

$$2x + y \leq 10 \tag{1}$$
$$x + 2y \leq 8 \tag{2}$$
$$x \geq 0 \tag{3}$$
$$y \geq 0 \tag{4}$$



Example 1.2.

$$\begin{aligned} 2x + y &> 10 && (5)\\ x + 2y &\geq 8 && (6)\\ x &\geq 0 && (7)\\ y &\geq 0 && (8) \end{aligned}$$



Example 1.3.

$$x + 4y \leq 32 \tag{9}$$
$$3x + y \leq 30 \tag{10}$$
$$4x + 5y \geq 51 \tag{11}$$

Example 1.4.

$$4x + 3y \leq 48 \tag{12}$$
$$2x + y \geq 24 \tag{13}$$
$$x \leq 9 \tag{14}$$

2. A Geometric Approach to Linear Programming in Two Dimensions

To illustrate the material, let us consider the following example:

Example 2.1.
A tent manufacturer makes a standard model and an expedition model. Each standard tent requires 1 labor-hour from the cutting department and 3 labor hours from the assembly department. Each expedition model requires 2 labor-hour from the cutting department and 4 labor hours from the assembly department. The per day maximum amount of labor hours for the cutting department is 32 labor-hours where the assembly department has 84 labor-hours available per day. The company makes a profit of \$50 on each standard tent it sells and a profit of \$80 on each expedition tent sold. Assuming that all manufactured tents will sell, how many tents of each type should be manufactured per day in order to maximize the total daily profit?

Solution:
We may summarize the information as follows:

Table 1. Data for Example 2.1

Labor-Hours	Standard Model	Expedition Model	Max per Day
Cutting Dept.	1	2	32
Assembly Dept.	3	4	84
Profit per Tent	\$50	\$80	

The first step will be to identify the ***decision variables*** for the model. In this situation, let us put

x_1 = the number of standard tents manufactured per day and

x_2 = the number of expedition tents manufactured per day.

We next form our ***objective function***, that is the function we wish to optimize. In this situation our objective function is

$$P(x_1, x_2) = 50x_1 + 80x_2. \tag{15}$$

According to this function, profit can be made arbitrarily large by letting x_1 and/or x_2 grow without bound. Unfortunately, the situation has restrictions due to the amount of available labor-hours. Thus we have the following ***problem constraints***:

$$\text{(Cutting Department Constraints) } 1x_1 + 2x_2 \leq 32 \text{ and} \tag{16}$$
$$\text{(Assembly Department Constraints:) } 3x_1 + 4x_2 \leq 84. \tag{17}$$

As well, we have the ***nonnegative constraints*** that

$$x_1 \geq 0 \text{ and} \tag{18}$$
$$x_2 \geq 0. \tag{19}$$

These constraints are usually expressed with the single statement

$$x_1, x_2 \geq 0.$$

Thus the mathematical model for the problem we are considering is

$$\text{Maximize: } P(x_1, x_2) = 50x_1 + 80x_2 \tag{20}$$
$$\text{Subject to: } \quad x_1 + 2x_2 \leq 32 \tag{21}$$
$$3x_1 + 4x_2 \leq 84 \tag{22}$$
$$x_1, x_2 \geq 0. \tag{23}$$

The graph of this system of linear inequalities given by the constraints is known as the ***feasible region***.

Now it is a wonderful thing that we are able to graph the feasible region and thus know the set of solutions to the system of linear inequalities, but which pair maximizes the profit function? This is a daunting task as there are infinitely many possible points (unless we only consider integer solutions; more on this later).

Our aim is to maximize $P(x_1, x_2) = 50x_1 + 80x_2$ so let us consider this function. If we fix a value for the profit, call it K, then we have a linear equation in two variables. In particular, if we solve for x_2 we have

$$x_2 = -\frac{5}{8}x_1 + \frac{K}{80}. \tag{24}$$

Notice that as K increases, this line moves further away from the origin. Recall that K is the profit. So, in order to maximize the profit but meet the conditions of the problem's constraints we should:

(Your answer here:)

In general, we may conclude that the optimal solution to a linear programming problem occurs at:

Though we have not formally proved it, we have

Theorem 2.2 (The Fundamental Theorem of Linear Programming)**.**
If the optimal value of the objective function in a linear programming problem exists, then that value (known as the optimal solution*) must occur at one or more of the corner points of the feasible region.*

Again, though we have not formally proved it we also have

Theorem 2.3.
If the feasible region of a linear programming problem is bounded, then

there exists a maximum and a minimum value for the objective function.

If the feasible region of a linear programming problem is unbounded and if the coefficients of the objective function are positive, then there exists a minimum value for the objective function, but there does not exist a maximum value for this function.

If the feasible region is empty, then there does not exist a maximum or a minimum value for the objective function.

Given this, let us answer the question in the previous example.

TABLE 2. $P(x_1, x_2)$ Evaluate at the Corner Points

Corner Point (x_1, x_2)	$P(x_1, x_2)$
$(0, 0)$	
$(28, 0)$	
$(20, 6)$	
$(0, 16)$	

Note that it is possible that the optimal solution occur at more than one corner point. If this situation occurs, then *any point of the line segment joining the corner points is also an optimal solution.*

3. SUMMARY

We may summarize our techniques as follows:

(1) Summarize the data in table form (see Table 1).
(2) Form a mathematical model for the problem by
  - introducing decision variables,
  - stating the objective function,
  - listing the problem constraints, and
  - writing the nonnegative constraints.
(3) Graph the feasible region.
(4) Make a table listing the value of the objective function at each corner point.
(5) The optimal solutions will be the largest and smallest values of the in this table.

4. Keywords

- *solution set*
- *bounded*
- *unbounded*
- *corner point*
- *decision variables*
- *objective function*
- *problem constraints*
- *nonnegative constraints*
- *feasible region*
- *optimal solution*

5. The Simplex Method (with problem constraints of the form "$\leq$")

5.1. Introduction.

We have thus far looked at a geometric means of solving a linear programming problem. Our previous method works fine when we have two unknowns and are thus working with a feasible region that is a subset of the real plane (in other words, is a two-dimensional object). This will also work with three variables, but the graphs of the feasible regions are much more complicated (they would be three dimensional objects). Of course, things get terribly messy if we have four or more unknowns (we could not draw their graphs). As such we must develop another means of tackling linear programming problems.

Our approach will begin with considering what we did in high school algebra when we were faced with a system of linear *equalities* of more than one variable. In particular, if we were faced with solving the system

$$\left\{\begin{array}{rrrr} 3x_1 & -5x_2 & +x_3 & = 17 \\ -2x_1 & +x_2 & -x_3 & = 5 \\ x_1 & -x_2 & +3x_3 & = -4 \end{array}\right\}.$$

We could isolate a variable, substitute and work to reduce the system to two equations in two unknowns, but that is too much work. Instead, we introduce an augmented matrix

$$\left[\begin{array}{rrrcr} 3 & -5 & 1 & | & 17 \\ -2 & 1 & -1 & | & 5 \\ 1 & -1 & 3 & | & -4 \end{array}\right]$$

and do Gauss-Jordan Elimination (row operations) to get the matrix in reduced form. It is this method that we adapt to solve linear programming problems.

5.2. Slack Variables.

The careful reader will have observed that in linear programming problems, the constraints are expressed as *inequalities* and not *equalities*. Hence we introduce ***slack variables***.

Recall that in Example 2.1 we had the problem constraints that

$$1x_1 + 2x_2 \le 32 \text{ and} \tag{25}$$

$$3x_1 + 4x_2 \le 84. \tag{26}$$

We then introduce the nonnegative variables s_1 and s_2 to "pick up the slack", i.e.

$$1x_1 + 2x_2 + s_1 = 32 \text{ and} \tag{27}$$

$$3x_1 + 4x_2 + s_2 = 84. \tag{28}$$

Unfortunately, we are now in a situation where we have four unknowns and two equations. This means that we now have infinitely many solutions to the system (we do know something about these solutions, though; namely that we can fix two variables and express the other two variables as a function of the fixed variables). We get around this little problem of an infinite number of solutions by introducing *basic* and *nonbasic* variables.

Definition 5.1 (Basic and Nonbasic Variables)**.**
Basic variables *are chosen arbitrarily but with the restriction that there are exactly the same number of basic variables as there are equations. We then say that the remaining variables are* **nonbasic variables***.*

We may now present the idea of a basic solution. We put the nonbasic variables equal to 0 and then the solution of the resulting system of linear equations is called a ***basic solution***.

Let us revisit Example 2.1 with the inclusion of the slack variables. We then have the linear programming problem:

$$\text{Maximize:} \quad P(x_1, x_2) = 50x_1 + 80x_2 \tag{29}$$

$$\text{Subject to:} \quad x_1 + 2x_2 + s_1 = 32 \tag{30}$$

$$3x_1 + 4x_2 + s_2 = 84 \tag{31}$$

$$x_1, x_2, s_1, s_2 \ge 0. \tag{32}$$

Notice that we have included nonnegativity conditions on the slack variables in (32) since the original inequality involved nonnegative numbers and was of the form "$\le$".

5.3. **The Method.**

We begin by writing the model as an augmented matrix with the objective function as the last line:

$$\left[\begin{array}{cccccc|c} (basic) & x_1 & x_2 & s_1 & s_2 & P & \\ (s_1) & 1 & 2 & 1 & 0 & 0 & 32 \\ (s_2) & 3 & 4 & 0 & 1 & 0 & 84 \\ \hline (P) & -50 & -80 & 0 & 0 & 1 & 0 \end{array}\right]$$

Note that each row refers to a **basic variable**, namely, in the above tableau, the first row (with numbers) corresponds to the basic variable s_1, the second row to the basic variable s_2, and the last row to the basic variable P.

Recall that our aim is to maximize profit. Since the expedition model of the tent produces the most profit, it seems reasonable that letting x_2 be as large as possible will lead to maximizing the profit. x_2 corresponds to the second column and notice that this column has the largest negative entry in last row (the row corresponding to the objective function). We will choose to work with this column and refer to it as the ***pivot column***.

Highlight 5.2 (Selecting the PIVOT COLUMN).
To select the pivot column*, choose the column with the largest negative entry in the bottom row. If there is a tie, choose either column. If there are no negative entries, we are done and an optimal solution has been found.*

Since we are focusing on making as many expedition models as possible, let us recall the constraints. In particular, we know that the cutting department needs 2 labor-hours to cut each tent and that the assembly department needs 4 labor-hours to assemble each tent. The cutting department only has 32 labor-hours available which means they can cut for at most 16 tents. The assembly department has only 84 labor-hours available, so they can assemble at most 21 tents. Hence we have the restriction that we can make at most 16 expedition models of the tent. Hence we choose the first row as the ***pivot row***. Notice that these restrictions correspond to *dividing the each value in the last column by the corresponding value in the pivot column and then selecting the smallest positive ratio.*

Highlight 5.3 (Selecting the PIVOT ROW).
To select the pivot row*, choose the row with the small positive ratio of the entry in the last column divided by the corresponding entry in the pivot column. If there is a tie, choose either column. If there are*

no positive entries in the pivot column above the last row, the linear program has no optimal solution and we are done.

We refer to the element of the tableau that is in the pivot row and the pivot column as the ***pivot element***. In our example, the pivot element is 2. Our job is to now perform legal row operations to make the pivot element 1 and every other entry in the pivot column 0. These row operations are commonly called ***pivot operations***.

Highlight 5.4 (PIVOT OPERATIONS)**.**
There are two:

- *Multiply the pivot row by the reciprocal of the pivot element. This transforms the pivot element into a 1. Symbolically, if k is the pivot element and R_t is the pivot row: $\frac{1}{k}R_t \to R_t$.*
- *Add multiples of the pivot row to all other rows in the tableau in order to annihilate (transform to 0) all other entries in the pivot column. Symbolically, something like $aR_s + R_t \to R_t$.*

Back to the example, we have decided that the second column (the column representing the variable x_2) is the pivot column and the first row (currently representing the basic variable s_1) is the pivot row. Hence we should begin our row operations by doing

$$\frac{1}{2}R_1 \to R_1$$

which gives us:

$$\left[\begin{array}{llllll|l} (basic) & x_1 & x_2 & s_1 & s_2 & P & \\ (s_1) & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 16 \\ (s_2) & 3 & 4 & 0 & 1 & 0 & 84 \\ \hline (P) & -50 & -80 & 0 & 0 & 1 & 0 \end{array}\right].$$

Before we continue, note that the basic variables are given by the set $\{s_1, s_2, P\}$ and the non-basic variables are $\{x_1, x_2, P\}$.

Next, we do the row operations

$$-4R_1 + R_2 \to R_2 \text{ and } 80R_1 + R_3 \to R_3$$

which gives us

$$\left[\begin{array}{llllll|l} (basic) & x_1 & x_2 & s_1 & s_2 & P & \\ (x_1) & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 16 \\ (s_2) & 1 & 0 & -2 & 1 & 0 & 20 \\ \hline (P) & -10 & 0 & 40 & 0 & 1 & 1280 \end{array}\right].$$

Observe that we now have that the basic variables are given by the set $\{x_1, s_2, P\}$ and the non-basic variables are $\{s_1, x_2, P\}$. It is also worthwhile to point out that we currently have $x_1 = 16$ and $x_2 = 0$ (since it is currently a non-basic variable). This corresponds to a corner point in the graph of the feasible solution and, in fact, the Simplex Method after one iteration has moved us from the origin to the point $(16, 0)$.

Since we still have a negative entry in the bottom row, we repeat the process. Column 1 will now be the pivot column. As well, $16/\frac{1}{2} = 32$ and $20/1 = 1$, hence we choose the second row as the pivot row. Our pivot element is already 1 so we do the row operations

$$-\frac{1}{2}R_2 + R_1 \rightarrow R_1 \text{ and } 10R_2 + R_3 \rightarrow R_3$$

which gives us

$$\left[\begin{array}{lccccc|c} (basic) & x_1 & x_2 & s_1 & s_2 & P & \\ (x_1) & 0 & 1 & \frac{3}{2} & 0 & 0 & 6 \\ (x_2) & 1 & 0 & -2 & 1 & 0 & 20 \\ \hline (P) & 0 & 0 & 20 & 10 & 1 & 1480 \end{array}\right].$$

Again, notice that x_2 has now entered as a basic variable. As well, since there are no more negative entries in the bottom row, the Simplex Method terminates and we have found the optimal solution; in particular that producing 6 standard model tents (which is what the variable x_1 represents) and producing 20 expedition model tents (represented by x_2) produces a maximum daily profit of \$1480.

5.4. **Summary.** To summarize the Simplex Method:

(1) Introduce slack variables into the mathematical model and write the initial tableau.
(2) Are there any negative entries in the bottom row?
    - Yes - go to step 3.
    - No - the optimal solution has been found.
(3) Select the pivot column.
(4) Are there any positive elements above the last row (above the solid line)?
    - Yes - go to step 5.
    - No - no optimal solution exists.
(5) Select the pivot row and thus the pivot element. Perform the appropriate pivot operations then return to step 2.

5.5. Keywords:

- *slack variables*
- *basic variables*
- *nonbasic variables*
- *basic solution*
- *pivot row*
- *pivot column*
- *pivot element*
- *pivot operation*

6. The Dual - Minimization with Problem Constraints of the form $\geq$

6.1. Introduction.

We are now interested in solving problems of the following form:

$$\text{Minimize:} \quad C(y_1, y_2) = 32y_1 + 84y_2 \tag{33}$$

$$\text{Subject to:} \quad y_1 + 3y_2 \geq 50 \tag{34}$$

$$2y_1 + 4y_2 \geq 80 \tag{35}$$

$$y_1, y_2 \geq 0. \tag{36}$$

As we will shortly see, each problem of this form can be associated with a corresponding maximization problem which we refer to as the ***dual problem***.

Our first step in forming the dual problem will be to form a matrix from the problem constraints and the objective function. The appropriate matrix for our example is

$$A = \left[\begin{array}{cc|c} 1 & 3 & 50 \\ 2 & 4 & 80 \\ \hline 32 & 84 & 1 \end{array}\right].$$

Please note that A is not the matrix associated with the initial simplex tableau. We now form A^T, the ***transpose*** of matrix A. The transpose of a matrix is formed by interchanging the matrix's rows and columns. Hence, for our example:

$$A^T = \left[\begin{array}{cc|c} 1 & 2 & 32 \\ 3 & 4 & 84 \\ \hline 50 & 80 & 1 \end{array}\right].$$

Given A^T we may now form the dual problem, namely we now have a maximization problem with constraints of the form $\leq$. In particular, we have the dual problem:

$$\text{Maximize: } P(x_1, x_2) = 50x_1 + 80x_2 \tag{37}$$
$$\text{Subject to: } x_1 + 2x_2 \leq 32 \tag{38}$$
$$3x_1 + 4x_2 \leq 84 \tag{39}$$
$$x_1, x_2 \geq 0. \tag{40}$$

Look familiar? This is the example we have considered previously. This time, though, let us name our slack variables y_1 and y_2. Hence the simplex method gives us:

$$\left[\begin{array}{ccccc|c} x_1 & x_2 & y_1 & y_2 & P & \\ 0 & 1 & \frac{3}{2} & 0 & 0 & 6 \\ 1 & 0 & -2 & 1 & 0 & 20 \\ \hline 0 & 0 & 20 & 10 & 1 & 1480 \end{array}\right].$$

We note that the bottom row of the simplex tableau gives us the solution to our minimization problem, namely that $y_1 = 20$ and $y_2 = 10$ minimizes $C(y_1, y_2) = 32y_1 + 84y_2$ under the stated constraints. I.e.,

Highlight 6.1.
An optimal solution to a minimization problem is obtained from the bottom row of the final simplex tableau for the dual maximization problem.

One word of caution, though.

Warning! 6.2.
Never multiply the inequality representing a problem constraint in a maximization problem by a number if that maximization problem is being used to solve a corresponding minimization problem.

We close by stating the following very important theorem without proof:

Theorem 6.3 (The Fundamental Principle of Duality)**.**
A minimization problem has a solution if and only if the corresponding dual maximization problem has a solution.

6.2. **Keywords:**

- *dual problem*
- *transpose of a matrix*