HOMEWORK #3 SOLUTIONS

Problem 1.
#1.42 *Let $A = \{$ January, February, $\ldots$, December $\}$. Given $x \in A$, let $f(x)$ be the number of days in x. Does $f(x)$ define a function from A to $\mathbb{N}$?*

Solution.
Note that $f($ February $) = 28$ or 29 depending on whether we are in a leap year or not, hence f is not a function from A to $\mathbb{N}$.

Problem 2.
#1.45 *Determine whether the rules below define functions from $\mathbb{R}$ to $\mathbb{R}$.*

Solution.
a) $f(x) := \begin{cases} |x-1| & \text{for } x < 4, \\ |x| - 1 & \text{for } x > 2. \end{cases}$
Notice there is overlap for $2 < x < 4$ and hence we must check these values. But for $x \in (2,4)$, $|x-1| = |x| - 1$ thus $f(x)$ is a function from $\mathbb{R}$ to $\mathbb{R}$.

b) $f(x) := \begin{cases} |x-1| & \text{for } x < 2, \\ |x| - 1 & \text{for } x > -1. \end{cases}$
Let $x = -0.9$. Then $f(-0.9)$ equals either $|-0.9-1| = 1.9$ or $|-0.9| - 1 = -0.1$, hence $f(x)$ is not a function from $\mathbb{R}$ to $\mathbb{R}$.

c) $f(x) := \begin{cases} \frac{(x+3)^2-9}{x} & \text{if } x \neq 0, \\ 6 & \text{if } x = 0. \end{cases}$
There is no overlap in the conditions defining the piece-wise function hence $f(x)$ is a function from $\mathbb{R}$ to $\mathbb{R}$.

d) $f(x) := \begin{cases} \frac{(x+3)^2-9}{x} & \text{if } x > 0, \\ x + 6 & \text{if } x < 7. \end{cases}$
Note that $\frac{(x+3)^2-9}{x} = x + 6$ as long as $x \neq 0$, hence the pieces agree over $(0,7)$ and $f(x)$ is thus a function from $\mathbb{R}$ to $\mathbb{R}$.

e) a) $f(x) := \begin{cases} \sqrt{x^2} & \text{for } x \geq 2, \\ x & \text{if } 0 \leq x \leq 4, \\ -x & \text{for } x < 0. \end{cases}$
Notice $\sqrt{x^2} = x$ for $x \in [2,4]$ hence $f(x)$ is a function from $\mathbb{R}$ to $\mathbb{R}$.

Problem 3.
#1.46 *Determine the images of the functions* $f : \mathbb{R} \to \mathbb{R}$ *defined as follows:*

a) $f(x) = \frac{x^2}{1+x^2}$ has $[0, 1)$ as its image.
b) $f(x) = \frac{x}{1+|x|}$ has $(-1, 1)$ as its image.

Problem 4.
#1.49 *Let* f *and* g *be functions from* $\mathbb{R}$ *to* $\mathbb{R}$. *Prove or disprove the following:*

a) f and g bounded implies $f + g$ is bounded.

Proof.
f and g bounded means that there exist M and $N \in \mathbb{R}$ such that $|f(x)| \le M$ and $|g(x)| \le N$. Thus, by the triangle inequality

$$|(f + g)(x)| := |f(x) + g(x)| \le |f(x)| + |g(x)| \le M + N$$

and thus $f + g$ is bounded. □

Note: to simplify things we henceforth write f instead of $f(x)$ and as well write g instead of $g(x)$.

b) f and g bounded imply fg is bounded.

Proof.
We begin by noting that

$$(a + b)^2 = a^2 + 2ab + b^2 \tag{1}$$

can be rewritten as

$$ab = \frac{1}{2}\left((a + b)^2 - a^2 - b^2\right) \tag{2}$$

f and g bounded means that there exist M and $N \in \mathbb{R}$ such that $|f| \le M$ and $|g| \le N$. By part a) we have $|f + g| \le M + N$. Hence

$$\begin{aligned} |fg| &= |f + g|^2 - |f|^2 - |g|^2 \text{ by (2)} \\ &\le |f + g|^2 + |f|^2 + |g|^2 \text{ since } |f|, |g| \ge 0 \\ &\le (M + N)^2 + M^2 + N^2. \end{aligned}$$

Hence fg is bounded. □

c) $f + g$ bounded implies f and g are bounded.

Proof.
This is not true. For a counterexample let $f(x) = x$ and let $g(x) = -x$. Then $(f+g)(x) = 0$ which is, of course, bounded but neither $f(x)$ nor $g(x)$ are bounded. □

d) fg bounded implies f and g are bounded.

Proof.
This is not true. For a counterexample let $f(x) = x$ and let $g(x) = \frac{1}{x}$. Then $(f+g)(x) = 1$ which is, of course, bounded but $f(x)$ is not bounded. □

e) $f+g$ and fg bounded imply f and g are bounded.

Proof.
$f+g$ bounded means $|f+g| \leq M$ and fg bounded means $|fg| \leq N$ for some M and $N \in \mathbb{R}$. Rewriting (1) gives us

$$f^2 + g^2 = (f+g)^2 - 2fg.$$

Hence

$$\begin{aligned}
|f|^2 \leq |f|^2 + |g|^2 &\leq |f+g|^2 - 2|fg| \\
&\leq |f+g|^2 + 2|fg| \\
&\leq M^2 + 2N.
\end{aligned}$$

Taking square roots,

$$|f| \leq \sqrt{M^2 + 2N}$$

shows f is bounded. The same argument shows g is bounded. □

Problem 5.
#1.50 *For S in the domain of f, define $f(S) := \{f(s) : s \in S\}$. Show that for C and D subsets of the domain of f that*

$$f(C \cap D) \subseteq f(C) \cap f(D).$$

Do we have equality?

Proof.
($\subseteq$) Let $y \in f(C \cap D)$. Then $\exists s \in C \cap D$ such that $y = f(s)$. But $s \in C \cap D$ means that $s \in C$ and $s \in D$, hence $y \in f(C)$ and $y \in f(D)$; i.e. $y \in f(C) \cap f(D)$, showing $f(C \cap D) \subseteq f(C) \cap f(D)$.
($\not\supseteq$). Let $f(x) = x^2$. Put $C = \{r \in \mathbb{R} : r < 0\}$ and put $D = \{r \in \mathbb{R} : r > 0\}$. Then $f(C) = f(D) = \mathbb{R}_{\geq 0}$, hence $f(C) \cap f(D) = \mathbb{R}_{\geq 0}$. But $C \cap D = \emptyset$, thus $f(C \cap D) \not\supseteq f(C) \cap f(D)$. □

Problem 6.
#1.51 *For* $f : A \to B$ *and* $S \subseteq B$*, define* $I_f(S) := \{a \in A : f(a) \in S\}$. *Note that* $I_f(S)$ *is called the* **preimage of** S *Let* $X, Y \subseteq B$.

a) Prove or disprove $I_f(X \cup Y) = I_f(X) \cup I_f(Y)$.
($\subseteq$) Let $a \in I_f(X \cup Y)$. Then $f(a) \in (X \cup Y)$. Thus $f(a) \in X$ or $f(a) \in Y$.
Case 1: Suppose $f(a) \in X$. Then, by definition of I_f, $a \in I_f(X)$, hence $a \in I_f(X) \cup I_f(Y)$ showing $I_f(X \cup Y) \subseteq I_f(X) \cup I_f(Y)$.
Case 2: Suppose $f(a) \in Y$. Then, by definition of I_f, $a \in I_f(X)$, hence $a \in I_f(X) \cup I_f(Y)$ showing $I_f(X \cup Y) \subseteq I_f(X) \cup I_f(Y)$.
($\supseteq$) Let $a \in I_f(X) \cup I_f(Y)$. Then $a \in I_f(X)$ or $a \in I_f(Y)$.
Case 1: Suppose $a \in I_f(X)$. Then $f(a) \in X$ and thus $f(a) \in X \cup Y$. Hence $a \in I_f(X \cup Y)$ and thus $I_f(X \cup Y) \supseteq I_f(X) \cup I_f(Y)$.
Case 2: Suppose $a \in I_f(Y)$. Then $f(a) \in X$ and thus $f(a) \in X \cup Y$. Hence $a \in I_f(X \cup Y)$ and thus $I_f(X \cup Y) \supseteq I_f(X) \cup I_f(Y)$.
Hence $I_f(X \cup Y) = I_f(X) \cup I_f(Y)$

b) Prove or disprove $I_f(X \cap Y) = I_f(X) \cap I_f(Y)$.
($\subseteq$) Let $a \in I_f(X \cap Y)$. Then $f(a) \in (X \cap Y)$. Thus $f(a) \in X$ and $f(a) \in Y$, i.e. $a \in I_f(X) \cap I_f(Y)$ showing $I_f(X \cap Y) \subseteq I_f(X) \cap I_f(Y)$.
($\supseteq$) Let $a \in I_f(X) \cap I_f(Y)$. Then $a \in I_f(X)$ and $a \in I_f(Y)$ which means $f(a) \in X$ and $f(a) \in Y$. Hence $f(a) \in (X \cap Y)$ thus $a \in I_f(X \cap Y)$ showing $I_f(X \cap Y) \supseteq I_f(X) \cap I_f(Y)$.
Since we have shown containment both ways $I_f(X \cap Y) = I_f(X) \cap I_f(Y)$.
$\square$

Problem 7.
#1.55 *Let* $\mathbb{F}$ *be a field consisting of exactly three elements* $0, 1, x$. *Obtain the addition and multiplication tables for* $\mathbb{F}$.

Solultion:
Before we begin let us state the following. Since $\mathbb{F}$ is a field we know $\exists u$, v, and $w \in \mathbb{F}$ such that $1 + u = 0$, $x + v = 0$, and $x \cdot w = 1$ (the existence of inverses).

0 is the additive identity and 1 the multiplicative identity, so there is nothing to show in these cases. We merely have to determine the following:
$\underline{x \cdot x :}$
Note that since $x \neq 0$, $x \cdot x \neq 0$. Hence $x \cdot x = 1$ or x. But if $x \cdot x = x$

then

$$\begin{aligned} x \cdot x &= x \\ x \cdot x \cdot w &= x \cdot w \\ x \cdot 1 &= 1 \\ x &= 1 \end{aligned}$$

which is impossible since $\mathbb{F}$ has three elements (in particular, $x \neq 1$). Hence we must have

$$x \cdot x = 1.$$

Before continuing, note that $\mathbb{F}$ is a commutative group under $+$, hence for all $a, b \in \mathbb{F}, a + b = b + a$.

$\underline{1 + x = x + 1:}$

Suppose $1 + x = 1$. Then

$$\begin{aligned} 1 + x &= 1 \\ 1 + x + u &= 1 + u \\ 1 + u + x &= 0 \text{ (since } \mathbb{F} \text{ is commutative under } +) \\ 0 + x &= 0 \\ x &= 0 \end{aligned}$$

which is a contradiction since $\mathbb{F}$ has three elements (i.e. $x \neq 0$). Suppose $1 + x = x$. Then

$$\begin{aligned} 1 + x &= x \\ 1 + x + v &= x + v \\ 1 + x + v &= 0 \\ 1 + 0 &= 0 \\ 1 &= 0 \end{aligned}$$

which is a contradiction since $\mathbb{F}$ has three elements (i.e. $1 \neq 0$). Thus

$$1 + x = 0 = x + 1. \tag{3}$$

$\underline{x + x:}$

Suppose $x + x = 0$. Then

$$\begin{aligned} x + x &= 0 \\ x + x + 1 &= 0 + 1 \\ x + 0 &= 1 \text{ [by (3)]} \\ x &= 1 \end{aligned}$$

which is a contradiction since $\mathbb{F}$ has three elements (i.e. $x \neq 1$).
Suppose $x + x = x$. Then

$$\begin{aligned} x + x &= x \\ x + x + v &= x + v \\ x + 0 &= 0 \\ x &= 0 \end{aligned}$$

which is a contradiction since $\mathbb{F}$ has three elements (i.e. $x \neq 0$). Thus

$$x + x = 1.$$

$\underline{1 + 1:}$
Suppose $1 + 1 = 0$. Then

$$\begin{aligned} 1 + 1 &= 0 \\ 1 + 1 + x &= 0 + x \\ 1 + 0 &= x \text{ [by (3)]} \\ 1 &= x \end{aligned}$$

which is a contradiction since $\mathbb{F}$ has three elements (i.e. $1 \neq x$).
Suppose $1 + 1 = 1$. Then

$$\begin{aligned} 1 + 1 &= 1 \\ 1 + 1 + u &= 1 + u \\ 1 + 0 &= 0 \\ 1 &= 0 \end{aligned}$$

which is a contradiction since $\mathbb{F}$ has three elements (i.e. $1 \neq 0$). Thus

$$1 + 1 = x.$$

So we have:

Table 1. Addition Table for $\mathbb{F}$

$+$	0	1	x
0	0	1	x
1	1	x	0
x	x	0	1

and

TABLE 2. Multiplication Table for $\mathbb{F}$

$\cdot$	0	1	x
0	0	0	0
1	0	1	x
x	0	x	1

Problem 8.
#1.56 *Does there exist a field $\mathbb{F}$ such that $|\mathbb{F}| = 4$? Does there exist a field $\mathbb{F}$ such that $|\mathbb{F}| = 6$?*

Solultion:

Problem 9.
#2.9 *Negate the statement "No slow learners attend this school."*

Solultion:
Note that the statement is equivalent to
"If someone attends this school, then they are not a slow learner."
We need the following lemma:

Lemma 10.
$\neg(p \to q) \equiv p \wedge \neg q$.

Proof:

TABLE 3. Truth Table for $\neg(p \to q) \equiv p \wedge \neg q$.

p	q	$\neg q$	$p \to q$	$\neg(p \to q)$	$p \wedge \neg q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

$\square$

Hence our statement is equivalent to
"Someone attends this school and is a slow learner"; i.e.
"There is at least one slow learner that attends this school" which is c).

Problem 11.
#2.14 *We define a circle C by*

$$C := \left\{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 + ax + by = c \text{ where } c > -\frac{a^2+b^2}{4} \right\}.$$

Thus C is completely specified by the parameters a, b, and c.

(1) *Using this definition, give examples of two circles such that*
   (a) *the circles do not intersect.*
   (b) *the circles intersect in exactly one common element.*
   (c) *the circles intersect in two common elements.*
(2) *Explain why the parameter c is restricted as given.*

Solultion:
We begin by completing the square on the equation that defines a circle, namely:

$$\begin{aligned} x^2 + y^2 + ax + by = c &\Leftrightarrow x^2 + ax + y^2 + by = c \\ &\Leftrightarrow x^2 + ax + \frac{a^2}{4} + y^2 + ay + \frac{b^2}{4} = c + \frac{a^2}{4} + \frac{b^2}{4} \\ &\Leftrightarrow \left(x + \frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 = c + \frac{a^2+b^2}{4} \qquad (4) \end{aligned}$$

Recall from Geometry that a circle C has as its equation

$$(x - \alpha)^2 + (y - \beta)^2 = r^2$$

where the center of C is given by (α, β) and the radius by r. Thus, by (4), the center of the circle as stated in the definition has

$$\text{center } \left(-\frac{a}{2}, -\frac{b}{2}\right) \text{ and radius } \sqrt{c + \frac{a^2+b^2}{4}}. \qquad (5)$$

Hence the easiest way to answer (1)(a) is to have two circles with the same center (i.e. ***concentric circles***) but different radii, hence let C_1 have center $(0,0)$ and radius 1 and let C_2 have center $(0,0)$ and radius

2. Thus for each of C_1 and C_2, $a = b = 0$ but for C_1

$$\begin{aligned}\sqrt{c + \frac{a^2 + b^2}{4}} &= 1, \text{ thus since } a = b = 0,\\ c + \frac{0}{4} &= 1, \text{ i.e.}\\ c &= 1.\end{aligned}$$

Whereas for C_2

$$\begin{aligned}\sqrt{c + \frac{a^2 + b^2}{4}} &= 2, \text{ thus since } a = b = 0,\\ c + \frac{0}{4} &= 4, \text{ i.e.}\\ c &= 4.\end{aligned}$$

So C_1 is given by $a = 0$, $b = 0$, $c = 1$ and C_2 is given by $a = 0$, $b = 0$, $c = 4$.

One way to answer (1)(b) is to have a circle, C_1, with center $(0, 0)$ and $r = 1$ and a second circle, C_2 with center $(2, 0)$. Clearly the circles intersect at exactly one place, namely $C_1 \cap C_2 = \{(1, 0)\}$. For C_1, $a = b = 0$ and $c = 1$. For C_2, clearly $b = 0$ but $-\frac{a}{2} = 2$ therefore $a = -4$. Since $r^2 = c + \frac{a^2+b^2}{4}$, $1 = c + \frac{16+0}{4}$, i.e. $c = -3$. Hence two circles that meet this condition are the circle given by the parameters $a = b = 0$, $c = 1$ and the circle given by the parameters $a = -4$, $b = 0$, and $c = -3$.

A way to answer (1)(c) is to consider the circle with center at $(0, 0)$ with $r = 2$ and the circle with center at $(3, 0)$ with $r = 2$. Since the centers each lie on the x-axis and are 3 units apart, the circles intersect at exactly two places (look at the graphs). The first circle is given by the parameters $a = b = 0$ with $c = 4$. Regarding the second circle, $b = 0$ but $-\frac{a}{2} = 3$ hence $a = -6$. Since $r^2 = c + \frac{a^2+b^2}{4}$, $4 = c + \frac{(-6)^2+0}{4}$, i.e. $c = -5$. Thus the second circle is given by the parameters $a = -6$, $b = 0$, and $c = -5$.

Regarding part (2), the left hand side of (4) is the sum of two squares and therefore is ≥ 0. In particular, the circle cannot have a radius of 0 (a circle with $r = 0$ is just a point), thus $c + \frac{a^2+b^2}{4} > 0$, establishing the desired relationship.

Problem 12.
#2.25 *For* $a \in \mathbb{R}$ *and* $f : \mathbb{R} \to \mathbb{R}$, *show that* (*a*) *and* (*b*) *have different*

meanings:

$$a)(\forall\epsilon)(\exists\delta)[(|x-a|<\delta)\Rightarrow(|f(x)-f(a)|<\epsilon)]$$
$$b)(\exists\delta)(\forall\epsilon)[(|x-a|<\delta)\Rightarrow(|f(x)-f(a)|<\epsilon)]$$

Solultion:
a) Hopefully you recognize this statement as the definition of continuity. The difference between a) and b) is that in a) we can find some δ for any given ϵ whereas in b) the claim is that there is some δ out there such that the statement $[(|x-a|<\delta)\Rightarrow(|f(x)-f(a)|<\epsilon)]$ is true for all ϵ. This second statement can only be true if $f(x)=c$ where $c\in\mathbb{R}$.

Problem 13.
#2.44 *Show that each of*

(1) $(q\wedge\neg q)\Rightarrow p$,
(2) $(p\wedge q)\Rightarrow p$, *and*
(3) $p\Rightarrow(p\vee q)$

are true (i.e. are **tautologies***).*

Proof.

Table 4. Truth Table for $(q\wedge\neg q)\Rightarrow p$.

p	q	$\neg q$	$q\wedge\neg q$	$(q\wedge\neg q)\Rightarrow p$
T	T	F	F	T
T	F	T	F	T
F	T	F	F	T
F	F	T	F	T

Table 5. Truth Table for $(p\wedge q)\Rightarrow p$.

p	q	$p\wedge q$	$(p\wedge q)\Rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

TABLE 6. Truth Table for $p \Rightarrow (p \vee q)$.

p	q	$p \vee q$	$p \Rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

□

Problem 14.
#2.47 *Let $P(x)$ be the statement "x is odd" and let $Q(x)$ be the statement "$x^2 - 1$ is divisible by 8". Prove or disprove*

(1) $(\forall x \in \mathbb{Z})\,[P(x) \Rightarrow Q(x)]$ *and*
(2) $(\forall x \in \mathbb{Z})\,[Q(x) \Rightarrow P(x)]$.

Before we prove the result, a Lemma:

Lemma 15.
For all $k \in \mathbb{Z}$, $k(k+1)$ is divisible by 2.

Proof of Lemma:
k and $k+1$ are consecutive integers, therefore exactly one of them is an even integer. □

Proof.
(1) This statement is true. Suppose $x \in \mathbb{Z}$ is odd. Then $\exists k \in \mathbb{Z}$ such that $x = 2k + 1$. Then

$$x^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k(k+1).$$

By Lemma (15), $k(k+1)$ is divisible by 2, therefore $4 \cdot 2 = 8$ is a factor of $x^2 - 1$.
(2) We prove the contrapositive, namely $\neg P(x) \Rightarrow \neg Q(x)$.
Suppose $x \in \mathbb{Z}$ is not odd; i.e. x is even. Then $\exists k \in \mathbb{Z}$ such that $x = 2k$. Thus $x^2 - 1 = (2k)^2 - 1 = 4k^2 - 1$.
Claim: $4k^2 - 1$ is not divisible by 8.
Proof of claim:
There are two cases:
Case 1: Suppose k is even.

Then $\exists m \in \mathbb{Z}$ such that $k = 2m$. Then $4k^2 - 1 = 4(2m)^2 - 1 = 16m^2 - 1$. 8 divides 16, but not 1, therefore $8 \nmid (16m^2 - 1)$.
Case 2: Suppose k is odd.
Then $\exists m \in \mathbb{Z}$ such that $k = 2m + 1$. Then $4k^2 - 1 = 4(2m+1)^2 - 1 = 4(m^2 + 2m + 1) - 1 = 4m^2 + 8m + 3$. 8 does not divide 3 therefore $8 \nmid (4m^2 + 8m + 3)$.

□

Problem 16.
#2.48 *Let $P(x)$ be the statement "x is odd" and let $Q(x)$ be the statement "$x = 2k$ for some $k \in \mathbb{Z}$". Prove or disprove*

(1) $(\forall x \in \mathbb{Z})\,[P(x) \Rightarrow Q(x)]$ *and*
(2) $(\forall x \in \mathbb{Z})\,(P(x)) \Rightarrow (\forall x \in \mathbb{Z})\,(Q(x))$.

Proof.
(1) is the statement "For all integers x, if x is odd, then $x = 2k$ for some $k \in \mathbb{Z}$". This is clearly false.
(2) is the statement "if all integers are odd, then all integers are twice some other integer". The premise of this conditional statement is false, therefore the conditional statement has a truth value of "true". □

Problem 17.
#2.49 *Let* $S := \{x \in \mathbb{R} : x^2 > x + 6\}$ *and let* $T := \{x \in \mathbb{R} : x > 3\}$. *Interpret the following in words and determine whether they are true or false:*

(1) $T \subseteq S$.
(2) $S \subseteq T$.

Solultion:
(1) says that if a real number x is greater than 3 then $x^2 > x + 6$. This statement is true.

Proof.
We begin by noting that $x^2 > x + 6$ is equivalent to $x^2 - x - 6 > 0$. Note that $f(x) = x^2 - x - 6$ has its vertex at $x = \frac{-(-1)}{2 \cdot 1} = \frac{1}{2}$ and is a parabola that opens up. Hence $f(x)$ is a strictly increasing function for $x > \frac{1}{2}$. Since $3^2 - 3 - 6 = 0$, then for any $x > 3$, $x^2 - x - 6 > 0$, i.e. $T \subseteq S$. □

(2) says that if a real number x is such that $x^2 > x + 6$ then x must be greater than 3. Let $x = -3$. Then $9 = (-3)^2 > -3 + 6 = 3$. Hence the statement is false, i.e. $S \nsubseteq T$.

We prove exercise 2.51 before we prove 2.50.

Problem 18.
#2.51 *Using statements about membership, prove the following:*

(1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

The above properties are known as the *distributive laws for set operations*. We prove the statements using *membership tables* (these should remind you exactly of truth tables).

Proof.
Let 1 represent $x \in$ and 0 represent $x \notin$.

A	B	C	$B \cap C$	$A \cup B$	$A \cup C$	$A \cup (B \cap C)$	$(A \cup B) \cap (A \cup C)$
0	0	0	0	0	0	0	0
1	0	0	0	1	1	1	1
0	1	0	0	1	0	0	0
0	0	1	0	0	1	0	0
0	1	1	1	1	1	0	0
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Since the membership tables (which cover all possible cases) agree, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. $\square$

A	B	C	$B \cup C$	$A \cap B$	$A \cap C$	$A \cap (B \cup C)$	$(A \cap B) \cup (A \cap C)$
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	1	1	0	1	1	1
1	1	0	1	1	0	1	1
1	1	1	1	1	1	1	1

Since the membership tables agree, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.$\square$

We prove a few lemmas before proceeding to #2.50.

Lemma 19 (DeMorgan's first law)**.**
$\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Proof.
We again use a membership table:

A	B	$\overline{A}$	$\overline{B}$	$A \cap B$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
0	0	1	1	0	1	1
1	0	0	1	0	1	1
0	1	1	0	0	1	1
1	1	0	0	1	0	0

Since the entries in the membership table agree, $\overline{A \cap B} = \overline{A} \cup \overline{B}$. $\square$

Lemma 20.
$A - B = A \cap \overline{B}$.

Proof.

A	B	$\overline{B}$	$A - B$	$A \cap \overline{B}$
0	0	1	0	0
1	0	1	1	1
0	1	0	0	0
1	1	0	0	0

Since the entries in the membership table agree, $A - B = A \cap \overline{B}$. □

Lemma 21.
$\overline{\overline{A}} = A$.

Proof.

A	$\overline{A}$	$\overline{\overline{A}}$
0	1	0
1	0	1

Since the entries in the membership table agree, $\overline{\overline{A}} = A$. □

Problem 22.
#2.50 *Prove the following:*

(1) $\overline{A \cup B} = \overline{A} \cap \overline{B}$ *(this is DeMorgan's second law).*
(2) $A \cap \overline{A \cap B} = A - B$.
(3) $A \cap \overline{A \cap \overline{B}} = A \cap B$.
(4) $(A \cup B) \cap \overline{A} = B - A$.

Proof. (1)

A	B	$\overline{A}$	$\overline{B}$	$A \cup B$	$\overline{A \cup B}$	$\overline{A} \cap \overline{B}$
0	0	1	1	0	1	1
1	0	0	1	1	0	0
0	1	1	0	1	0	0
1	1	0	0	1	0	0

Since the entries in the membership table agree, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

(2)

$$\begin{aligned} A \cap \overline{A \cap B} &= A \cap \left(\overline{A} \cup \overline{B}\right) \text{ (DeMorgan's first law)} \\ &= \left(A \cap \overline{A}\right) \cup \left(A \cap \overline{B}\right) \text{ (distributive property)} \\ &= \emptyset \cup \left(A \cap \overline{B}\right) \\ &= A \cap \overline{B} \\ &= A - B \text{ (by Lemma 20).} \end{aligned}$$

(3)

$$\begin{aligned} A \cap \overline{A \cap \overline{B}} &= A \cap \left(\overline{A} \cup \overline{\overline{B}}\right) \text{ by DeMorgan's first law} \\ &= A \cap \left(\overline{A} \cup B\right) \text{ by Lemma 21} \\ &= \left(A \cap \overline{A}\right) \cup (A \cap B) \text{ by distributive property} \\ &= \emptyset \cup (A \cap B) \\ &= A \cap B \end{aligned}$$

(4)

$$\begin{aligned} (A \cup B) \cap \overline{A} &= \left(A \cap \overline{A}\right) \cup \left(B \cap \overline{A}\right) \text{ (by distributive property)} \\ &= \emptyset \cup \left(B \cap \overline{A}\right) \\ &= \left(B \cap \overline{A}\right) \\ &= B - A \text{ (by Lemma 20).} \end{aligned}$$

□

Problem 23.
#3.14 *Write each of the following sums in summation notation and find and prove a formula in terms of n for each:*

(1) $3 + 7 + 11 + \cdots + (4n - 1)$.
(2) $1 + 5 + 9 + \cdots + (4n + 1)$.
(3) $-1 + 2 - 3 + 4 - \cdots - (2n - 1) + 2n$.
(4) $1 - 3 + 5 - 7 + \cdots + (4n - 3) - (4n - 1)$.

Solultion:
(1) In summation notation,

$$3 + 7 + 11 + \cdots + (4n - 1) = \sum_{i=1}^{n} (4i - 1).$$

Put $S(n) = \sum_{i=1}^{n} (4n - 1)$. Note the following:

n	$S(n)$	$S(n) + n$	$\frac{S(n)+n}{n}$
1	3	4	$4 = 2 \cdot 2$
2	10	12	$6 = 2 \cdot 3$
3	21	24	$8 = 2 \cdot 4$
4	36	40	$10 = 2 \cdot 5$

So it seems that

$$\begin{aligned} \frac{S(n) + n}{n} &= 2(n + 1) \\ \frac{S(n) + n}{n} &= 2n + 2 \\ S(n) + n &= 2n^2 + 2n \\ S(n) &= 2n^2 + n = n(2n + 1). \end{aligned}$$

Doing a check: $S(1) = 1(2 \cdot 1 + 1) = 3$, $S(2) = 2(2 \cdot 2 + 1) = 10$, $S(3) = 3(2 \cdot 3 + 1) = 21$, and $S(4) = 4(2 \cdot 4 + 1) = 36$. We prove our claim by induction:

Proof.
Base Step:
We have already shown that $S(1) = 3$.

Inductive Step:
Suppose $S(n) = n(2n+1)$ (**induction hypothesis**). Then

$$\begin{aligned} S(n+1) &= \sum_{i=1}^{n+1}(4i-1) \\ &= \sum_{i=1}^{n}(4i-1) + 4[n+1] - 1 = \sum_{i=1}^{n}(4i-1) + 4n + 3 \\ &= n(2n+1) + 4n + 3 \text{ (by the induction hypothesis)} \\ &= 2n^2 + 5n + 3 \\ &= (2n+3)(n+1) \\ &= (2[n+1]+1)([n+1]). \end{aligned}$$

□

Note that another proof can be done by using results regarding the summation notation, namely:

$$\sum_{i=1}^{n}(4i-1) = 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = 4\left(\frac{n(n+1)}{2}\right) - n = 2n^2 + 2n - n = 2n^2 + n.$$

Solultion:
(2) In summation notation,

$$1 + 5 + 9 + \cdots + (4n+1) = \sum_{i=0}^{n}(4i+1).$$

Put $S(n) = \sum_{i=1}^{n}(4n+1)$. Note the following:

n	$S(n)$	$\frac{S(n)}{n+1}$
0	1	1
1	6	3
2	15	5
3	28	7

So it seems that

$$\frac{S(n)}{n+1} = 2n+1$$
$$S(n) = (n+1)(2n+1).$$

Doing a check: $S(1) = 1(2 \cdot 0 + 1) = 1$, $S(2) = 2(2 \cdot 1 + 1) = 6$, $S(3) = 3(2 \cdot 2 + 1) = 15$, and $S(4) = 4(2 \cdot 3 + 1) = 28$. We prove our claim by induction:

Proof.
Base Step:
We have already shown that $S(1) = 1$.
Inductive Step:
Suppose $S(n) = (n + 1)(2n + 1)$ (**induction hypothesis**). Then

$$\begin{aligned}
S(n+1) &= \sum_{i=0}^{n+1}(4i+1) \\
&= \sum_{i=0}^{n}(4i+1) + 4[n+1] + 1 = \sum_{i=0}^{n}(4i-1) + 4n + 5 \\
&= (n+1)(2n+1) + 4n + 5 \text{ (by the induction hypothesis)} \\
&= 2n^2 + 2n + n + 1 + 4n + 5 \\
&= 2n^2 + 7n + 6 \\
&= (2n+3)(n+2). \\
&= (2[n+1]+1)([n+1]+1)
\end{aligned}$$

□

Note that another proof can be done by using results regarding the summation notation, namely:

$$\begin{aligned}
\sum_{i=0}^{n}(4i+1) &= 1 + \sum_{i=1}^{n}(4i+1) \\
&= 1 + 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\
&= 1 + 4\left(\frac{n(n+1)}{2}\right) + n \\
&= 1 + 2n^2 + 2n + n \\
&= 2n^2 + 3n + 1 \\
&= (2n+1)(n+1).
\end{aligned}$$

(3) In summation notation,

$$-1+2-3+4-\cdots-(2n-1)+2n=\sum_{i=1}^{n}\left(-[2i-1]+2i\right).$$

But

$$\sum_{i=1}^{n}\left(-[2i-1]+2i\right)=\sum_{i=1}^{n}\left(-2i+1+2i\right)=\sum_{i=1}^{n}1=n.$$

So this sum is n.

(4) In summation notation,

$$1-3+5-7+\cdots+(4n-3)-(4n-1)=\sum_{i=1}^{n}\left([4n-3]-[4n-1]\right).$$

But

$$\sum_{i=1}^{n}\left([4n-3]-[4n-1]\right)=\sum_{i=1}^{n}\left(4n-3-4n+1\right)=\sum_{i=1}^{n}(-2)=-2n.$$

So this sum is $-2n$.