Infinite Sequences and Series

Calculus 2 (Wheeler) - The University of Pittsburgh
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MWF 9:00-9:50 (Course #10052) Room 426 Benedum
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1. Definitions and Basics

Definition 1.1 (Sequence)**.**
A sequence $\{a_n\}$ *is an ordered list of numbers.*

Theorem 1.2.
Let $\{a_n\}$ *be a sequence. If* $\lim_{n\to\infty} f(x) = L$ *and if* $f(n) = a_n$ *whenever* n *is a positive integer, then* $\lim_{n\to\infty} a_n = L$.

Note that by this theorem all of our limit laws for functions now apply to sequences. As well, we have a Squeeze Theorem for Sequences:

Theorem 1.3 (Squeeze Theorem for Sequences)**.**
If $a_n \leq b_n \leq c_n$ *for* $n \geq n_0$ *where* n_0 *is a positive integer and if* $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$*, then* $\lim_{n\to\infty} b_n = L$.

Definition 1.4.
If $a_n < a_{n+1}$ *for all* $n \geq 1$*, then* $\{a_n\}$ *is called an* increasing sequence*. As well, if* $a_n > a_{n+1}$ *for all* $n \geq 1$*, then* $\{a_n\}$ *is called an* decreasing sequence*. A sequence that is increasing or decreasing is called a* monotonic sequence

Definition 1.5.
If there exists a real number M *such that* $a_n \leq M$ *for all* $n \geq 1$*, then* $\{a_n\}$ *is said to be* bounded above*. If there exists a real number* N *such that* $a_n \geq N$ *for all* $n \geq 1$*, then* $\{a_n\}$ *is said to be* bounded below*. A sequence that is both bounded above and bounded below is said to be a* bounded sequence*.*

Theorem 1.6 (Monotonic Sequence Theorem)**.**
Every bounded, monotonic sequence is convergent.

Definition 1.7 (Series)**.**
Given a sequence $\{a_n\}$ *we can form the* series $\sum_{n=1}^{\infty} a_n$*. Let* $s_k = \sum_{n=1}^{k} a_n$ *be the* k^{th} *partial sum of the series. Then we define* $\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} s_k$*.*

Definition 1.8 (Geometric Series)**.**
A geometric series *is a series of the form*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots$$

This series converges if $|r| < 1$ *and diverges if* $|r| \geq 1$*. In particular,*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ if } |r| < 1.$$

Example 1.9 (Harmonic Series)**.**
The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges.*

Before continuing let us note that every series $\sum_{n=1}^{\infty} a_n$ has associated with it *two* sequences, namely $\{a_n\}$ and $\{s_k\}$ (the sequence of partial sums).

Theorem 1.10.
If the series $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Theorem 1.11 (The Test for Divergence)**.**
If $\lim_{n\to\infty} a_n \neq 0$*, then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Theorem 1.12.
If c *is a constant and if* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent series, then* $\sum_{n=1}^{\infty} ca_n$, $\sum_{n=1}^{\infty}(a_n + b_n)$*, and* $\sum_{n=1}^{\infty}(a_n - b_n)$ *are also convergent series. Moreover*

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n,$$

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n, \text{ and}$$

$$\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n.$$

2. Convergence Tests for Series

Name	Conditions	Test
Integral Test	f continuous, positive, decreasing on $[1, \infty)$ with $f(n) = a_n$	$\int_1^\infty f(x)dx$ converges $\Rightarrow \sum_{n=1}^\infty a_n$ converges $\int_1^\infty f(x)dx$ diverges $\Rightarrow \sum_{n=1}^\infty a_n$ diverges
p-Series		$\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$
Comparison Test	a_n and b_n are positive	$a_n \leq b_n$, $\sum b_n$ convergent $\Rightarrow \sum a_n$ convergent $a_n \geq b_n$, $\sum b_n$ divergent $\Rightarrow \sum a_n$ divergent where the inequality is true for all $n \geq N$.
Limit Comparison Test	a_n and b_n are positive	If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a positive finite number, then either both series converge or both diverge
Alternating Series Test	$b_n > 0$ $b_{n+1} \leq b_n$ for all n $\lim_{n\to\infty} b_n = 0$	$\sum_{n=1}^\infty (-1)^{n-1} b_n = b_1 - b_2 + \cdots$ converges
Absolute Convergence		$\sum_{n=1}^\infty \lvert a_n \rvert$ converges $\Rightarrow \sum_{n=1}^\infty a_n$ coverges
Ratio Test		$\lim_{n\to\infty} \lvert\frac{a_{n+1}}{a_n}\rvert < 1 \Rightarrow \sum_{n=1}^\infty \lvert a_n \rvert$ converges $\lim_{n\to\infty} \lvert\frac{a_{n+1}}{a_n}\rvert > 1 \Rightarrow \sum_{n=1}^\infty a_n$ diverges $\lim_{n\to\infty} \lvert\frac{a_{n+1}}{a_n}\rvert = 1 \Rightarrow$ nothing
Root Test		$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} < 1 \Rightarrow \sum_{n=1}^\infty \lvert a_n \rvert$ converges $\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} > 1 \Rightarrow \sum_{n=1}^\infty a_n$ diverges $\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} = 1 \Rightarrow$ nothing

3. A Note on Rearranging Infinite Sums

One must be careful when dealing with infinity. As an example, consider the following series. It can be shown that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2. \tag{1}$$

Multiply both sides by $\frac{1}{2}$ to get

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \cdots = \frac{1}{2}\ln 2.$$

Inserting zeros we get

$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \cdots = \frac{1}{2}\ln 2. \tag{2}$$

Adding equations (1) and (2) we get

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2}\ln 2. \tag{3}$$

A careful observation will yield that equations (1) and (3) have the same terms but different sums!

It turns out that

Theorem 3.1.
If $\sum a_n$ is absolutely convergent with sum s, then any rearrangement of $\sum a_n$ has the same sum s.

Before we state a result related to the example we just considered, we need the following definition:

Definition 3.2 (Conditionally Convergent Series)**.**
A series $\sum a_n$ is said to be conditionally convergent *if it is convergent but is not absolutely convergent.*

We may now state a result due to Georg Reimann.

Theorem 3.3.
If $\sum a_n$ is a conditionally convergent series and r is any real number, then there is a rearrangement of $\sum a_n$ that has a sum equal to r.

4. Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where the c_i's are constants (called the **coefficients** of the power series) and x is a variable. Since we are considering a series, it makes sense to talk of when the power series converges and when it diverges. Further, note that a power series is a function of x, i.e.

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where the domain of $f(x)$ is the set of all x for which the power series converges.

We have already seen an example of a power series, namely

Example 4.1.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$$

which converges when $|x| < 1$*, i.e. when* x *is in the interval* $(-1, 1)$.

We refer to the interval $(-1, 1)$ as the *interval of convergence.*

More generally, we consider **power series centered at** a or **power series about** a (some also say a **power series in** $(x-a)$):

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where we adopt the convention that $(x-a)^0 = 1$ when $x = a$. Hence a power series always converges when $x = a$.

Theorem 4.2.
For a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ *there are only three possibilities:*

(1) *The series converges only when* $x = a$,
(2) *The series converges for all* x, *or*
(3) *There exists a positive number* R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$.

The R in the theorem is known as the **radius of convergence**. Note that the theorem says nothing if $|x-a| = R$, hence we must check these values for convergence. In particular, in (3) we have four possibilities for the interval of convergence, namely $(a - R, a + R)$, $[a - R, a + R)$, $(a - R, a + R]$, or $[a - R, a + R]$.