1. Homework 3.3 #6/4.1 #10

We wish to find a formula for

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}.$$

The first values are

$$\begin{aligned}
&(n=1)\ \frac{1}{1\cdot 2}=\frac{1}{2}\\
&(n=2)\ \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{1}{2}+\frac{1}{6}=\frac{2}{3}\\
&(n=3)\ \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}=\frac{3}{4}\\
&(n=4)\ \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}=\frac{4}{5}\\
&\qquad\vdots
\end{aligned}$$

A reasonable guess seems to be

$$\sum_{i=1}^{n}\frac{1}{i(i+1)}=\frac{n}{n+1}.$$

We can verify our guess by Induction:

Proof.
The base step has been established in the above work.

Now suppose the statement is true for $n = k$ where $k \geq 1$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \frac{1}{2} + \frac{1}{6} + \cdot + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

□