NAME: __________________________ Math 0240-1040 Calculus 3
CIRCLE ONE: Grade my WORK / ANSWERS Exam 2

Answer each of the following. Please show all work (no work=no credit).

1. (15 points) Use partial derivatives and the Second Derivative Test to find the point(s) on the cone $z^2 = x^2+y^2$ closest to the point $(4, 2, 0)$. Use the Second Derivative Test to justify that your answer is in fact a minimum.

   Solution:
   We have

$$f(x,y) = d^2 = (x-4)^2 + (y-2)^2 + z^2 \tag{1}$$
$$= (x-4)^2 + (y-2)^2 + x^2 + y^2 \tag{2}$$

   Minimizing $f(x,y) = d^2$ will minimize d, thus

$$f_x = 2(x-4) + 2x = 4x - 8 \text{ and} \tag{3}$$
$$f_y = 2(y-2) + 2y = 4y - 4. \tag{4}$$

   Thus the critical point is $(2, 1)$. Further,

$$f_{xx} = 4, \tag{5}$$
$$f_{yy} = 4, \text{ and} \tag{6}$$
$$f_{xy} = 0. \tag{7}$$

   Since $D(2,1) = 4 \cdot 4 - 0 = 16 > 0$ and since $f_{xx}(2,1) = 4 > 0$, by the Second Derivative Test, $f(x,y)$ has a minimum at $(2,1)$. When $x = 2$ and $y = 1$, $z^2 = 2^2 + 1^2$, i.e. $z = \pm\sqrt{5}$.
   Therefore the points on $z^2 = x^2 + y^2$ closest to the point $(4, 2, 0)$ are $(2, 1, \sqrt{5})$ and $(2, 1, -\sqrt{5})$.

2. (15 points) Use Lagrange Multipliers to find the point(s) on the cone $z^2 = x^2 + y^2$ closest to the point $(4, 2, 0)$.

   Solution:
   Minimize $f(x,y) = d^2 = (x-4)^2 + (y-2)^2 + z^2$ subject to $z^2 = x^2 + y^2$, i.e. $0 = x^2 + y^2 - z^2$ (so $g(x,y) = x^2 + y^2 - z^2$).
   Hence

$$\nabla f = \langle 2x-8, 2x-4, 2z \rangle \text{ and} \tag{8}$$
$$\nabla g = \langle 2x, 2y, -2z \rangle \tag{9}$$

   Thus we seek to solve $\lambda f = \lambda \nabla g$, i.e. we seek solutions to the system

$$2x - 8 = 2\lambda x \tag{10}$$
$$2y - 4 = 2\lambda y \tag{11}$$
$$2z = -2\lambda z \tag{12}$$

   Solving equation (12) for λ gives us $\lambda = \frac{2z}{-2z} = -1$. Substituting this value for λ into equations (10) and (11) gives us that $x = 2$ and $y = 1$. Plugging these values back into the original equation gives $z = \pm\sqrt{5}$. We note that this must be a minimum value for $f(x,y)$ since $f(x,y)$ is never negative and grows without bound. Thus the points on $z^2 = x^2 + y^2$ closest to the point $(4, 2, 0)$ are $(2, 1, \sqrt{5})$ and $(2, 1, -\sqrt{5})$.

3. (19 points) Find the mass and the center of mass of the lamina that occupies the region $D = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 2\}$ with density function $\rho(x, y) = 3xy$.

Solution:

$$\begin{aligned} m &= \int_0^2 \int_0^4 3xy \; dx \; dy && (13) \\ &= \int_0^2 24y \; dy = 48, && (14) \end{aligned}$$

$$\begin{aligned} M_x &= \int_0^2 \int_0^4 3xy^2 \; dx \; dy && (15) \\ &= \int_0^2 24y^2 \; dy = 64, && (16) \end{aligned}$$

and

$$\begin{aligned} M_y &= \int_0^2 \int_0^4 3x^2y \; dx \; dy && (17) \\ &= \int_0^2 64y^2 \; dy = 128. && (18) \end{aligned}$$

So

$$\overline{x} = \frac{M_y}{m} = \frac{128}{48} = \frac{8}{3}$$

and

$$\overline{y} = \frac{M_x}{m} = \frac{64}{48} = \frac{4}{3}.$$

Thus $(\overline{x}, \overline{y}) = (\frac{8}{3}, \frac{4}{3})$.

4. (15 points) Set up, but do not evaluate, the double integral that gives the surface area of the helicoid (spiral ramp) with vector equation $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle$ where $0 \leq u \leq 1$ and $0 \leq v \leq \pi$.

Solution:

$$\begin{aligned} \mathbf{r}_u &= \langle \cos v, \sin v, 0 \rangle && (19) \\ \mathbf{r}_v &= \langle -u \sin v, u \cos v, 1 \rangle && (20) \end{aligned}$$

so

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{vmatrix} && (21) \\ &= \begin{vmatrix} \sin v & 0 \\ u \cos v & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \cos v & 0 \\ -u \sin v & 1 \end{vmatrix} \mathbf{j} \begin{vmatrix} \cos v & \sin v \\ -u \sin v & u \cos v \end{vmatrix} \mathbf{k} && (22) \\ &= \sin v \mathbf{i} - \cos v \mathbf{j} + (u \cos^2 v + u \sin^2 v) \mathbf{k} && (23) \\ &= \langle \sin v, -\cos v, u \rangle. && (24) \end{aligned}$$

Hence

$$\begin{aligned} |\mathbf{r}_u \times \mathbf{r}_v| &= \sqrt{\sin^2 v + \cos^2 v + v^2} && (25) \\ &= \sqrt{1 + v^2}. && (26) \end{aligned}$$

So the surface area is

$$A(S) = \int_0^{\pi} \int_0^1 \sqrt{1 + v^2} \; du \; dv.$$

5. (18 points) Use a triple integral to find the volume of the solid enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 16$.
Solution:

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^4 \int_{r^2}^{16} r \; dz \; dr \; d\theta && (27) \\ &= \int_0^{2\pi} \int_0^4 (16r - r^3) \; dr \; d\theta && (28) \\ &= \int_0^{2\pi} \left[8r^2 - \frac{r^4}{4}) \right]_0^4 \; d\theta && (29) \\ &= \int_0^{2\pi} 64 \; d\theta && (30) \\ &= 128\pi. && (31) \end{aligned}$$

6. (18 points) Evaluate

$$\int\int\int_H (x^2 + y^2) dV$$

where H is the hemispherical region that lies above the xy-plane and below the sphere $x^2 + y^2 + z^2 = 1$.
Solution:
Since $(x^2 + y^2) = r^2 = (\rho \sin\phi)^2$,

$$\begin{aligned} \int\int\int_H (x^2 + y^2) dV &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 (\rho \sin\phi)^2 \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta && (32) \\ &= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin^3\phi \; d\phi \int_0^1 \rho^4 \; d\rho && (33) \\ &= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin\phi(1 - \cos^2\phi) \; d\phi \int_0^1 \rho^4 \; d\rho && (34) \\ &= 2\pi \cdot \left[-\cos\phi + \frac{1}{3}\cos^3\phi \right]_0^{\frac{\pi}{2}} \cdot \frac{1}{5} && (35) \\ &= \frac{4\pi}{15}. && (36) \end{aligned}$$