KEY **Math 115-009 Calculus 1**

CIRCLE ONE: Grade my WORK / ANSWERS **Exam 1**

Answer each of the following. Please show all work (no work=no credit). As well, if you use a named theorem to solve a problem, you are expected to state the name of the theorem.

Find the following derivatives:

1. (10 points)

$$\lim_{x\to-\infty} 3 = 3.$$

2. (10 points)

$$\begin{aligned}\lim_{t\to 2}\frac{t^2-4}{t^3-8} &= \lim_{t\to 2}\frac{(t-2)(t+2)}{(t-2)(t^2+2t+4)}\\ &= \lim_{t\to 2}\frac{t+2}{t^2+2t+4}\\ &= \frac{2+2}{2^2+2\cdot 2+4}\\ &= \frac{4}{12} = \frac{1}{3}.\end{aligned}$$

3. (10 points) $\lim_{x\to\pi}(x-\pi)^2\sin(\frac{1}{x-\pi})$

Solution:

Note that $-1 \le \sin\frac{1}{x-\pi} \le 1$ for $x \ne \pi$. Therefore

$$\begin{aligned}\lim_{x\to\pi} -1(x-\pi)^2 \le \lim_{x\to\pi}(x-\pi)^2\sin(\frac{1}{x-\pi}) &\le \lim_{x\to\pi} 1(x-\pi)^2\\ \text{so}\qquad 0 \le \lim_{x\to\pi}(x-\pi)^2\sin(\frac{1}{x-\pi}) &\le 0.\end{aligned}$$

Hence, by the Squeeze Theorem,

$$\lim_{x\to\pi}(x-\pi)^2\sin(\frac{1}{x-\pi}) = 0.$$

4. Let $h(x) = \frac{2x^2-8}{x^2-3x+2}$.

(a) (3 points) What are $h(1)$ and $h(2)$?

Note that $h(x) = \frac{2x^2-8}{x^2-3x+2} = \frac{2(x-2)(x+2)}{(x-2)(x-1)}$. Hence, $h(1)$ and $h(2)$ are undefined.

(b) (8 points) Use limits to find the vertical asymptote(s) of $h(x)$.

$$\begin{aligned}\lim_{x\to 1^+}\frac{2x^2-8}{x^2-3x+2} &= \lim_{x\to 1^+}\frac{2(x-2)(x+2)}{(x-2)(x-1)}\\ &= \lim_{x\to 1^+}\frac{2(x+2)}{x-1}\\ &= +\infty.\end{aligned}$$

Or

$$\begin{aligned}\lim_{x\to 1^-}\frac{2x^2-8}{x^2-3x+2} &= \lim_{x\to 1^-}\frac{2(x-2)(x+2)}{(x-2)(x-1)}\\ &= \lim_{x\to 1^-}\frac{2(x+2)}{x-1}\\ &= -\infty.\end{aligned}$$

Hence $x = 1$ is a vertical asymptote.

But

$$\begin{aligned}
\lim_{x\to 2^-} \frac{2x^2-8}{x^2-3x+2} &= \lim_{x\to 2^-} \frac{2x^2-8}{x^2-3x+2} \\
&= \lim_{x\to 2} \frac{2x^2-8}{x^2-3x+2} \\
&= \lim_{x\to 2} \frac{2(x-2)(x+2)}{(x-2)(x-1)} \\
&= \lim_{x\to 2} \frac{2(x+2)}{x-1} \\
&= \frac{2(2+2)}{2-1} \\
&= 8.
\end{aligned}$$

Hence there is no vertical asymptote at $x = 2$.

(c) (8 points) Use limits to find the horizontal asymptote(s) of $h(x)$.

$$\lim_{x\to\infty} \frac{2x^2-8}{x^2-3x+2} = \lim_{x\to-\infty} \frac{2x^2-8}{x^2-3x+2} = 2.$$

Hence $y = 2$ is a horizontal asymptote.

(d) (3 points) Sketch a graph of $h(x)$.

5. Let $f(x) = \begin{cases} 2x - x^2 & \text{if } x \le 2 \\ 2 - x & \text{if } 2 < x \le 3 \\ x - 4 & \text{if } 3 < x < 4 \\ \pi & \text{if } x \ge 4. \end{cases}$

(a) (12 points) For each of the numbers $2, 3$, and 4, determine whether $f(x)$ is continuous from the left, continuous from the right, or continuous at the number. Your work must justify your answer.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x - x^2) = 2(2) - 2^2 = 0.$$

But

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2 - x) = 2 - 2 = 0.$$

Therefore

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} f(x) = 0.$$

Moreover

$$f(2) = 2 \cdot 2 - 2^2 = 0.$$

So, since $\lim_{x \to 2} f(x) = f(2)$, $f(x)$ is continuous at 2.

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (2 - x) = 2 - 3 = -1.$$

But

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x - 4) = 3 - 4 = -1.$$

Therefore

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = -1.$$

Moreover

$$f(3) = 2 - 3 = -1.$$

So, since $\lim_{x \to 3} f(x) = f(3)$, $f(x)$ is continuous at 3.

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (x - 4) = 4 - 4 = 0.$$

But

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \pi = \pi.$$

Since

$$\lim_{x \to 4^-} f(x) \neq \lim_{x \to 4^+} f(x),$$

$\lim_{x \to 4} f(x)$ does not exist and therefore $f(x)$ is not continuous at 4.

(b) (3 points) Sketch a graph of $f(x)$.

6. (8 points) Use the *Intermediate Value Theorem* to show that there is a root of the equation $e^{-x^2} = x$ in the interval $(0,1)$.
Solution:
Let $f(x) = e^{-x^2} - x$. Then $e^{-x^2} = x$ has a root exactly when $f(x) = 0$. Notice $f(0) = 1$ but $f(1) = \frac{1}{e} - 1 < 0$. Hence, since $f(x)$ is continuous, by the *Intermediate Value Theorem* there exists a c in $(0,1)$ such that $f(c) = 0$, i.e. there is c in $(0,1)$ such that $e^{-c^2} = c$.

7. (7 points) Determine whether $k(x) = \frac{x^2+1}{x^3-x}$ is even, odd, or neither. Your work must justify your answer.
Solution:

$$\begin{aligned} k(-x) &= \frac{(-x)^2+1}{(-x)^3-(-x)} \\ &= \frac{x^2+1}{-x^3+x} \\ &= -\frac{x^2+1}{x^3-x} \\ &= -k(x). \end{aligned}$$

Hence $k(x)$ is an odd function.

8. Consider the curve $f(x) = 5 - x^2$.
   (a) (10 points) Use the definition of the derivative to find $f'(x)$.
   Solution:

$$
\begin{aligned}
f'(x) &:= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{5-(x+h)^2-(5-x^2)}{h} \\
&= \lim_{h\to 0} \frac{5-(x^2+2xh+h^2)-(5-x^2)}{h} \\
&= \lim_{h\to 0} \frac{5-x^2-2xh-h^2-5+x^2)}{h} \\
&= \lim_{h\to 0} \frac{-2xh-h^2}{h} \\
&= \lim_{h\to 0} \frac{h(-2x-h)}{h} \\
&= \lim_{h\to 0} (-2x-h) \\
&= -2x.
\end{aligned}
$$

(b) (2 points) What is the slope of the line tangent to the curve at $(2, 1)$?
Solution:

$$m_{(\text{tangent line at} x=2)} = f'(2) = -2(2) = -4.$$

(c) (4 points) Find an equation of this tangent line.
Solution:

$$
\begin{aligned}
y &= y_0 + m(x - x_0) \\
&= 1 + -4(x-2) \\
&= -4x + 9.
\end{aligned}
$$

(d) (2 points) Graph the curve $y = 5 - x^2$ and the tangent line at (2,1).
Solution: