Math 115 – 09 (Calculus I)

Duquesne University

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Pascal's Triangle and the Binomial Theorem

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1 Background

Recall that Pascal's Triangle has the form:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & \vdots & & & \vdots & & & \vdots &
\end{array}
$$

This is useful for expanding binomials, in particular:

Example 1.1

$$
\begin{aligned}
(2x-3y)^4 &= 1(2x)^4(-3y)^0 + 4(2x)^3(-3y)^1 + 6(2x)^2(-3y)^2 + 4(2x)^1(-3y)^3 + 1(2x)^0(-3y)^4 \\
&= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4.
\end{aligned}
$$

But why does this work?

2 Binomial Coefficients

You have most likely seen $C(n,r) = \binom{n}{r} := \frac{n!}{r!(n-r)!}$ where $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$. Note that we define $0! = 1$. These are known as the ***binomial coefficients*** and are used to count the number of ways to select r objects (without replacement) from a group of n objects.

Notice that $\binom{1}{1} = 1$, $\binom{2}{0} = 1$, $\binom{2}{1} = 2$, $\binom{2}{2} = 1$, $\binom{3}{0} = 1$, $\binom{3}{1} = 3$, ... Do these numbers look familiar?

Our observations make in reasonable to conjecture that Pascal's Triangle is really

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{2} & & \binom{4}{4} \\
 & \vdots & & & \vdots & & & \vdots &
\end{array}
$$

3 The Binomial Theorem

Our conjecture can be proven and is, in fact, known as the ***Binomial Theorem***:

Theorem 3.1 (The Binomial Theorem)
For and nonnegative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

4 Two Proofs

4.1 Proof by Induction on n

This involves using

Lemma 4.1 (Pascal's Identity)
For a nonnegative integer k and a positive integer j,

$$\binom{k+1}{j} = \binom{k}{j-1} + \binom{k}{j}.$$

Proof (of Pascal's Identity).
By definition,

$$\begin{aligned}\binom{k}{j-1}+\binom{k}{j} &:= \frac{k!}{(k-[j-1])!(j-1)!}+\frac{k!}{(k-j)!j!}\\ &= \frac{k!j}{(k-j+1)!j(j-1)!}+\frac{k!(k-j+1)}{(k-j+1)(k-j)!j!}\\ &= \frac{k!j+k!(k-j+1)}{(k-j+1)!j!}\\ &= \frac{k![j+k-j+1]}{(k-j+1)!j!}\\ &= \frac{k![k+1]}{(k-j+1)!j!}\\ &= \frac{(k+1)!}{(k+1-j)!j!}\\ &= \binom{k+1}{j}.\end{aligned}$$

This completes the proof.

It is worthwhile to note why this result is called Pascal's Identity. Recall that the entries of Pascal's Triangle are, in fact, the binomial coefficients. The method one uses to generate the entries in the triangle is exactly given by this identity (to get the j^{th} entry in the $(k+1)^{st}$ row, add the $(j-1)^{st}$ and the j^{th} entry in the k^{th} row).

Now that we are equipped with this identity, we may prove the Binomial Theorem. Our proof is by induction on n.

Proof (of the Binomial Theorem).
(Base Step) Let $n = 0$. Then

$$1=(a+b)^0=\sum_{k=0}^{0}\binom{n}{k}a^{n-k}b^k=\binom{0}{0}a^0b^0=1.$$

Hence, the statement we are trying to prove is true for the base case $n = 0$.

(Inductive Step) Assume for $k \geq 0$ that

$$(a+b)^k=\sum_{m=0}^{k}\binom{k}{m}a^{k-m}b^m \quad \text{(this is the \textit{induction hypothesis})}.$$

Then

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)(a+b)^k \\
&= (a+b)\sum_{m=0}^{k}\binom{k}{m}a^{k-m}b^m \text{ (by the induction hypothesis)} \\
&= a\left(\sum_{m=0}^{k}\binom{k}{m}a^{k-m}b^m\right) + b\left(\sum_{m=0}^{k}\binom{k}{m}a^{k-m}b^m\right) \\
&= \sum_{m=0}^{k}\binom{k}{m}a^{k-m+1}b^m + \sum_{m=0}^{k}\binom{k}{m}a^{k-m}b^{m+1} \\
&= a^{k+1} + \binom{k}{1}a^k b^1 + \cdots + \binom{k}{k}ab^k + \binom{k}{0}a^k b^1 + \cdots + \binom{k}{k-1}ab^k + b^{k+1} \\
&\text{by combining like terms and using } \textit{Pascal's Identity} \\
&= \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k b^1 + \cdots + \binom{k+1}{k}ab^k + \binom{k+1}{k+1}b^{k+1} \\
&= \sum_{m=0}^{k+1}\binom{k+1}{m}a^{k+1-m}b^m.
\end{aligned}$$

Since we have shown that the statement being true for the integer k implies the statement is true for $k+1$, the Principle of Mathematical Induction establishes the Binomial Theorem. This completes the proof.

4.2 A Combinatorial Proof

This proof involves counting and is absolutely soul-inspiringly beautiful.
Consider multiplying $(a = b)^n = \underbrace{(a+b)(a+b)\cdots(a+b)}_{n \text{ factors}}$. The only way to get an a^n term when we multiply is to choose an a from each of the n factors. This is the same as choosing no $b's$. This can be done in $\binom{n}{0}$ many ways. Likewise, to get an a^{n-1} term, we would choose exactly one b when from then n binomials when multiplying. This can be done in $\binom{n}{1}$ ways. And so on. Thus

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.$$

This completes the proof.