1. Propositions

Definition 1.1 (Proposition)**.**
A ***proposition*** *is a declarative statement that is either true or false but not both.*

Examples of propositions are the following:
Today is Monday.
The moon is made of green cheese.

Whereas these are not propositions:
Study hard.
The Pittsburgh Steelers are the greatest football team ever.
This statement is false.
He is sad.
$x + 4 = 7$.

Propositions are usually denoted by lower case letters such as p, q, or r.

2. Propositional Calculus

The simplest operation that can be done on a proposition is **negation**. The truth table for the operation of negation is:

Table 1. Truth Table for Negation

p	$\neg p$
T	F
F	T

Another type of operation involves joining two or more propostions. Compound propositions can be formed by combining propositions with the words *and* and *or*. The operation *and* is called **conjunction** where *or* is called **disjunction**. Their truth tables are as follows:

TABLE 2. Truth Table for Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Note that the conjunction of two propositions is true when each of the propositions are two and false otherwise.

TABLE 3. Truth Table for Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Note that the disjunction of two propositions is true when at least one of the propositions is true.

The operation $\vee$ is often called the *inclusive or* since it regards the compound proposition as true even if both the individual propositions are true. This leads to the definition of the *exclusive or*, denoted by $\oplus$, where the compound proposition is regarded as false in the case where both propositions are true. The truth table for $\oplus$ is

TABLE 4. Truth Table for the Exclusive Or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Truth tables are a very useful tool as we will se soon. Before continuing, though, let us make the following observation:

Remark 2.1.
A truth table with n propositions has 2^n rows.

3. Conditional Statements

A *conditional statement* is a statement of the form "if-then". The standard example of a conditional statement is:

$$\text{If it is raining, then it is wet.} \tag{1}$$

If we let p be the antecedent ("it is raining") and q the consequent ("it is wet"), we can write (1) symbolically as

$$p \rightarrow q$$

and often read this statement as "p implies q".

We will point out that the "if" part is often called the **hypothesis** or the **antecedent** where the "then" part is usually called the **consequent** or **conclusion**. The conditional statement is defined by the following truth table:

Table 5. Truth Table for a Conditional Statement

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

There are some important forms of the conditional statement $p \rightarrow q$. They are as follows:

The **converse** is the statement $q \rightarrow p$.
The **inverse** is the statement $\neg p \rightarrow \neg q$.
The **contrapositive** is the statement $\neg q \rightarrow \neg p$.

The statement "p if and only if q" is called the **biconditional** and its truth table is:

Table 6. Truth Table for a Biconditional Statement

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

4. Equivalence of Logical Statements

Logical statements are considered equivalent in they have the same truth table. As an example we prove the following:

Theorem 4.1.

$$\neg p \vee q \equiv p \to q.$$

Proof.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the statements $\neg p \vee q$ and $p \to q$ have the same truth values, they are equivalent. □

5. Exercises

(1) Prove that the statement $p \to q$ is logically equivalent to its contrapositive.
(2) Show that $p \leftrightarrow q$ is logically equivalent to $(p \to q \wedge q \to p)$.
(3) Prove **DeMorgan's Laws**:
  (a) $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$
  (b) $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$